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ADVANCES IN MATHEMATICAL ANALYSIS AND ITS APPLICATIONS

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Chapter 1

Some applications of double sequences

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1.1 Introduction

In this chapter we discuss relationships between theory of real double sequences and selection principles theory, a field of mathematics having nice and deep relations with various mathematical disciplines: game theory, combinatorics, function spaces, and so on. By $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively. Single sequences will be denoted by $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ and so on, while double sequences will be denoted by $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$, $\mathbf{Y} = (y_{m,n})_{m,n\in\mathbb{N}}$ and so on. We use the symbol c_2 to denote the set of real double sequences; $c_{2,+}$ denotes the set of double sequences of positive real numbers.

1.1.1 Double sequences

In 1900, Alfred Israel Pringsheim introduced the concept of convergence of real double sequences:

1. a double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ (notation $\mathsf{P}\text{-}\lim \mathbf{X} = a$ or $\mathsf{P}\text{-}\lim x_{m,n} = a$), if

$$\lim_{\min\{m,n\}\to\infty} x_{m,n} = a,$$

i.e. if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $m, n > n_0$ (see [27], and also [14, 29]). The limit a is called the *Pringsheim limit* of $\mathbf{X}$.

In this chapter we denote by $c_2^{a,P}$ the set of all double real sequences converging to a point $a \in \mathbb{R}$ in Pringsheim's sense,and similarly for $c_{2,+}^{a,P}$.

We also consider the following two kinds of convergence of double sequences.

2. a double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ *max-converges* to $a \in \mathbb{R}$ (notation max-lim $\mathbf{X} = a$ or max-lim $x_{m,n} = a$), if

$$\lim_{\max\{m,n\}\to\infty} x_{m,n} = a,$$

i.e. if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $m > n_0$ or for all $n > n_0$.

$c_2^{a,\max}$ denotes the class of real double sequences such that $\lim_{\max\{m,n\}\to\infty} x_{m,n} = a$;

3. a double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ *sum-converges* to $a \in \mathbb{R}$ (notation sum-lim $\mathbf{X} = a$ or sum-lim $x_{m,n} = a$), if

$$\lim_{m+n\to\infty} x_{m,n} = a,$$

i.e. if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $m, n \in \mathbb{N}$ such that $m + n > n_0$.

$c_2^{a,\mathrm{sum}}$ denotes the set of real double sequences sum-converging to a.

Observe

$$c_2^{a,\mathrm{sum}} \subsetneq c_2^{a,\max} \subsetneq c_2^{a,P}.$$

The most investigated convergence of double sequence is P-convergence. A considerable number of papers which appeared in recent years study mostly the set $c_2^{a,P}$ and its subsets from different points of view (see, for instance, the papers [1, 10, 12, 19, 20, 21, 23, 24, 25, 30, 31] and the books [22, 32]). Some results in this investigation generalize known results concerning single sequences to certain classes of double sequences, while other results reflect a specific nature of the Pringsheim convergence (for example, a double sequence may converge without being bounded).

In [13], Hardy introduced the notion of *regular convergence* for double sequences: a double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ regularly converges to a point $a \in \mathbb{R}$ if it P-converges to a and for each $m \in \mathbb{N}$ and each $n \in \mathbb{N}$ there exist the following two limits:

$$\lim_{n\to\infty} x_{m,n} = R_m,$$
$$\lim_{m\to\infty} x_{m,n} = C_n.$$

$\overline{c}_2^{a,P}$ denotes the set of elements $(x_{m,n})_{m,n\in\mathbb{N}}$ in $c_2^{a,P}$ which are bounded, regular and such that $\lim_{m\to\infty} x_{m,n} = \lim_{n\to\infty} x_{m,n} = a$.

A double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ is *bounded* if there is $M > 0$ such that $|x_{m,n}| < M$ for all $m, n \in \mathbb{N}$.

Notice that a P-convergent double sequence need not be bounded.

If $\mathsf{P}\text{-}\lim|\mathbf{X}| = \infty$, (equivalently, for every $M > 0$ there are $n_1, n_2 \in \mathbb{N}$ such that $|x_{m,n}| > M$ whenever $m \geq n_1$, $n \geq n_2$), then $\mathbf{X}$ is said to be *definitely divergent*.

We give now a few facts which will be used in the sequel without special mention.

Fact 1. To each double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ the following single sequences are assigned:

1.1. the *Landau-Hurwicz sequence* $\omega(\mathbf{X}) = (\omega_n(\mathbf{X}))_{n\in\mathbb{N}}$, where for each $n \in \mathbb{N}$

$$\omega_n(\mathbf{X}) = \sup\{|x_{k,l} - x_{p,q}| : k \geq n, l \geq n, p \geq n, q \geq n\}.$$

1.2. the *diagonal sequence* $d(\mathbf{X}) = (d_n(\mathbf{X}))_{n\in\mathbb{N}}$, where

$$d_n(\mathbf{X}) = \sum_{k=1}^{n}\left(\sum_{l=1}^{n} x_{k,l}\right).$$

If there is $D(\mathbf{X}) \in \mathbb{R}$ such that $\lim_{n\to\infty} d_n(\mathbf{X}) = D(\mathbf{X})$, one says that $\mathbf{X}$ has the *finite diagonal sum*, denoted by $D\text{-}\Sigma\mathbf{X}$.

ℓ_2^D denotes the class of double sequences $\mathbf{X}$ from c_2 with finite diagonal sum $D\text{-}\Sigma\mathbf{X}$;

1.3. the sequence $v(\mathbf{X}) = (v_n(\mathbf{X}))n \in \mathbb{N}$, where

$$v_n(\mathbf{X}) = \sum_{i=1}^{n-1}(x_{i,n} + x_{n,i}) + x_{n,n}.$$

Fact 2. To each double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ one assigns the double sequence

$$\mathbf{S}(\mathbf{X}) \ = (s_{m,n}(\mathbf{X}))_{m,n\in\mathbb{N}},$$

where

$$s_{m,n}(\mathbf{X}) = \sum_{k=1}^{m}\left(\sum_{l=1}^{n} x_{k,l}\right).$$

If there is a number $T(\mathbf{X}) = P\!-\!\lim \mathbf{S}(\mathbf{X})$, then we say that $\mathbf{X}$ has the *finite Pringsheim sum*, denoted by $P\text{-}\Sigma\mathbf{X}$.

ℓ_2^P denotes the class of double sequences $\mathbf{X} \in \mathbb{S}_2$ with finite Pringsheim sum P-$\Sigma\mathbf{X}$.

Fact 3. If we have a double sequence $(\mathbf{X}^{k,l})_{k,l\in\mathbb{N}}$ of double sequences $\mathbf{X}^{k,l} = (x_{m,n}^{k,l})_{m,n\in\mathbb{N}}$, then we can arrange it in a sequence $(\mathbf{X}^i = (x_{m,n}^i)_{m,n\in\mathbb{N}} : i \in \mathbb{N})$ of double sequences.

The following proposition shows a connection between P-converges of double sequences and their Landau-Hurwicz sequences.

Proposition 1 ([2]) *A double sequence* $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ *belongs to the class* $c_2^{a,P}$*,* $a \in \mathbb{R}$*, if and only if* $\lim_{n\to\infty} \omega_n(\mathbf{X}) = 0$.

Proof ($\Rightarrow$) Assume that then double sequence $\mathbf{X}$ belongs to $c_2^{a,P}$ for an arbitrary and fixed $a \in \mathbb{R}$. Let $\varepsilon > 0$ be given. There is $n_0 \in \mathbb{N}$ such that $|x_{j,k} - a| \leq \varepsilon/2$ for each $j \geq n_0$ and each $k \geq n_0$. Therefore we have

$$|x_{j,k} - x_{r,s}| = |x_{j,k} - a + a - x_{r,s}| \leq |x_{j,k} - a| + |x_{r,s} - a| \leq \varepsilon/2 + \varepsilon/2$$

for all $j, k, r, s \geq n_0$. This implies that for each $n \geq n_0$ we have

$$0 \leq \omega_n(\mathbf{X}) \leq \sup\{|x_{j,k} - x_{r,s}| : j \geq n_0, k \geq n_0, r \geq n_0, s \geq n_0\} \leq \varepsilon,$$

i.e. $\lim_{n\to\infty} \omega_n(\mathbf{X}) = 0$.

($\Leftarrow$) Let $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence with $\lim_{n\to\infty} \omega_n(\mathbf{X}) = 0$. For a given $\varepsilon > 0$, there is $n_1 \in \mathbb{N}$ such that $0 \leq |x_{j,k} - x_{r,s}| \leq \varepsilon/2$ for $j \geq n_1$, $k \geq n_1$, $r \geq n_1$, $s \geq n_1$, because

$$0 \leq \omega_n(\mathbf{X}) = \sup\{|x_{j,k} - x_{r,s}| : j \geq n_1, k \geq n_1, r \geq n_1, s \geq n_1\} \leq \varepsilon/2$$

for $n \geq n_1$. Since for all $j, r \geq n_1$ it holds $|x_{j,j} - x_{r,r}| \leq \varepsilon/2$, it follows that the sequence $(x_{t,t})$ is convergent (as a Cauchy sequence), i.e. there is $A \in \mathbb{R}$ such that $\lim_{t\to\infty} x_{t,t} = A$. This implies there is $n_2 \in \mathbb{N}$ such that $|x_{t,t} - A| \leq \varepsilon/2$ for each $t \geq n_2$. Therefore, for $n_0 = \max\{n_1, n_2\}$ and all $j, k \geq n_0$ we have

$$|x_{j,k} - A| \leq |x_{j,k} - x_{j,j}| + |x_{j,j} - A| \leq \varepsilon.$$

1.1.2 Selection principles

Selection principles theory is an old theory with roots in 1920s and 1930s. Nowadays it is one of the most investigated areas of mathematics. For more details concerning this theory see, for example, [15]. In this chapter we will discuss the selection principles related to collections of single or double sequences.

Let $\mathcal{A}$ and $\mathcal{B}$ be (not necessarily distinct) subfamilies of c_2. Then:

1. $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$

of elements in $\mathcal{A}$ there is a sequence $(a_n : n \in \mathbb{N})$ such that for each n, $a_n \in A_n$ and $(a_n : n \in \mathbb{N}) \in \mathcal{B}$ [15].

2. $\mathsf{S}_1^{(\mathsf{d})}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each double sequence $(\mathbf{A}_{m,n} : m, n \in \mathbb{N})$ of elements of $\mathcal{A}$ there are elements $a_{m,n} \in \mathbf{A}_{m,n}$ such that the double sequence $(a_{m,n})_{m,n \in \mathbb{N}}$ belongs to $\mathcal{B}$ [8].

3. Consider now an order on the set $\mathbb{N} \times \mathbb{N}$. Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Set $(m_1, n_1) \leq_\varphi (m_2, n_2) \Leftrightarrow \varphi(m_1, n_1) \leq \varphi(m_2, n_2)$, where $\leq$ is the natural order in $\mathbb{N}$.

$\mathsf{S}_1^{\varphi}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathbf{A}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $\mathbf{B} = (b_{\varphi^{-1}(n)})_{n \in \mathbb{N}}$ in $\mathcal{B}$ such that $b_{\varphi^{-1}(n)} \in \mathbf{A}_n$ for all $n \in \mathbb{N}$ [8].

4. $\alpha_2(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathbf{A}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $\mathbf{B}$ in $\mathcal{B}$ such that $\mathbf{B} \cap \mathbf{A}_n$ is infinite for all $n \in \mathbb{N}$ [16, 17].

5. $\alpha_2^{(d)}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each double sequence $(\mathbf{A}_{m,n} : m, n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $\mathbf{B}$ in $\mathcal{B}$ such that $\mathbf{B} \cap \mathbf{A}_{m,n}$ is infinite for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ [8].

Notice that to each of these selection principles one associates, in a natural way, an infinitely long two person game.

1.1.3 Asymptotic analysis

In [5] (see also [18] about asymptotic analysis of divergent processes) the class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})$ of translationally rapidly varying (single) sequences was introduced and studied: a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of positive real numbers belongs to the class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})$ of *translationally rapidly varying sequences* if

$$\lim_{n \to \infty} \frac{x_{[n+\alpha]}}{x_n} = 0$$

for each $\alpha \geq 1$. (Here $[r]$ denotes the integer part of $r \in \mathbb{R}$.)

By $\ell^D_{2,\mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})}$ we denote the subclass of ℓ^D_2 consisting of double sequences $\mathbf{X}$ such that $\omega(\mathbf{S}(\mathbf{X})) \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})$.

A generalization of this notion to double sequences is given in the following definition.

Definition 1 ([2]) A double sequence $\mathbf{X} = (x_{m,n})_{m,n \in \mathbb{N}} \in c_{2,+}$ belongs to the class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$ of *translationally rapidly varying double sequences* if

$$\lim_{\min\{m,n\} \to \infty} \frac{x_{[m+\alpha],[n+\beta]}}{x_{m,n}} = 0$$

for each $\alpha \geq 0$ and each $\beta \geq 0$ such that $\max\{\alpha, \beta\} \geq 1$. Here $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Example 1 The class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$ is nonempty. The double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ defined by

$$x_{m,n} = \frac{1}{(m+n)!}, \; m \in \mathbb{N}, n \in \mathbb{N}.$$

belongs to this class.

Proposition 2 ([2]) $\mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty}) \subset c^{0,P}_{2,+}$.

Proof Let $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$ and let $\varepsilon = \frac{1}{2}$, $\alpha = \beta = 1$. There is $n_0 = n_0(1/2, 1, 1) \in \mathbb{N}$ such that

$$\frac{x_{m+1,n+1}}{x_{m,n}} \le \frac{1}{2}$$

for all $m, n \ge n_0$. For $m = n \ge n_0$ we have $x_{n+1,n+1} \le \frac{1}{2} x_{n,n}$. Therefore, $\lim_{n\to\infty} x_{n,n} = 0$. Similarly, for $\varepsilon = \frac{1}{2}$ and $\alpha = 1$, $\beta = 0$, there is $n_1 = n_1(1/2, 1, 0) \in \mathbb{N}$ such that $x_{m+1,n} \le \frac{1}{2} x_{m,n}$ for all $m, n \ge n_1$ which implies that for $n \ge n_1$, $\lim_{m\to\infty} x_{m,n} = 0$. Finally, for $\varepsilon = \frac{1}{2}$, $\alpha = 0$, $\beta = 1$ there is $n_2 = n_2(1/2, 0, 1) \in \mathbb{N}$ such that $x_{m,n+1} \le \frac{1}{2} x_{m,n}$ for all $m, n \ge n_2$. From here we get $\lim_{n\to\infty} x_{m,n} = 0$, for each $m \ge n_2$.

Let now $\varepsilon > 0$ be arbitrary (and fixed). Then there is $n_\varepsilon \in \mathbb{N}$ such that $x_{n,n} \le \varepsilon$ for each $n \ge n_\varepsilon$. Set $n^* = \max\{n_\varepsilon, n_1, n_2\}$. Then $x_{m,n} \le \varepsilon$ for each $m, n \ge n^*$, which means that $\mathbf{X} \in c^{0,P}_{2,+}$.

The following example shows that the inclusion in the above proposition is proper.

Example 2 The double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ defined by

$$x_{m,n} = 1/m,$$

for m $\in \mathbb{N}$, $n \in \{1, 2, \cdots, m\}$,1/n, for $n \in \mathbb{N}$, $m \in \{1, 2, \cdots, n\}$.

evidently belongs to the class $c^{0,P}_{2,+}$. However, it does not belong to $\mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$ because for $\alpha = \beta = 1$ and $m = n$ we have

$$\lim_{n\to\infty} \frac{x_{m+1,n+1}}{x_{m,n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

1.2 S_1 selection principle and double sequences

In this section we present a few results which show applications of the selection principle S_1 to double sequences.

Theorem 1 ([8]) *For $a \in \mathbb{R}$ the selection principle $\mathsf{S}_1^{(\mathsf{d})}(c_2^{a,P}, \overline{c}_2^{a,P})$ is true.*

Proof Let $(\mathbf{X}^{j,k} : j,k \in \mathbb{N})$ be a double sequence of elements in $c_2^{a,P}$, and assume that for all $j,k \in \mathbb{N}$, $\mathbf{X}^{j,k} = (x^{j,k}_{m,n})_{m,n\in\mathbb{N}}$. Construct the double sequence $\mathbf{Y} = (y_{m,n})_{m,n\in\mathbb{N}}$ in the following way:

1. $y_{1,1} = x^{1,1}_{m_1,m_1} \in \mathbf{X}^{1,1}$, where $m_1 \in \mathbb{N}$ is such that $\left|x^{1,1}_{m,n} - a\right| \leq \frac{1}{2}$ for each $m \geq m_1$ and each $n \geq m_1$.

For $s,t \in \mathbb{N}$ and $q = \max\{s,t\} \geq 2$, we select $y_{s,t}$ to be $x^{s,t}_{m_p,m_p} \in \mathbf{X}^{s,t}$, where

$$p = (q-1)^2 + t,$$

if q = s, $(q-1)^2 + 2t - s$, if q = t,and $\left|x^{s,t}_{m,n} - a\right| \leq \frac{1}{2^q}$ for each $m \geq m_p$ and each $n \geq m_p$.

We prove $\mathbf{Y} = (y_{m,n})_{m,n\in\mathbb{N}} \in c_2^{a,P}$. Let $\varepsilon > 0$ be given. Choose $r \in \mathbb{N}$ such that $\frac{1}{2^r} < \varepsilon$. For each $m \geq r$ and each $n \geq r$, by construction of $\mathbf{Y}$, we have $|y_{m,n} - a| \leq \frac{1}{2^r} < \varepsilon$, i.e. $\mathbf{Y} \in c_2^{a,P}$. From the of $\mathbf{Y}$ we also easily conclude that $\mathbf{Y}$ actually belongs to $\overline{c}_2^{a,P}$. The theorem is proved.

Corollary 1 *For $a \in \mathbb{R}$ the selection principle $\mathsf{S}_1^{(\mathsf{d})}(c_2^{a,P}, c_2^{a,P})$ is satisfied.*

An improvement of this corollary is the following result.

Theorem 2 ([9]) *For a given $a \in \mathbb{R}$, the selection principle $\mathsf{S}_1^{(\mathsf{d})}(c_2^{a,P}, c_2^{a,\mathrm{sum}})$ holds.*

Proof Let $(\mathbf{X}^{j,k} = (x^{j,k}_{m,n})_{m,n\in\mathbb{N}} : j,k \in \mathbb{N})$ be a double sequence of elements in $c_2^{a,P}$. We define a sequence $\mathbf{Y} = (y_{j,k})_{j,k\in\mathbb{N}}$ in the following way.

For fixed $j,k \in \mathbb{N}$ pick an element $x^{j,k}_{m,n} \in \mathbf{X}^{j,k}$ such that $\left|x^{j,k}_{m,n} - a\right| \leq \left(\frac{1}{2}\right)^{j+k-1}$. In this way we get the double sequence $\mathbf{Y}$. We prove $\mathbf{Y} \in c_2^{a,\mathrm{sum}}$. For each $\varepsilon > 0$ find $i_0 \in \mathbb{N}$ so that $\frac{1}{2^i} < \varepsilon$ for all $i \geq i_0$. Then for all $j,k \in \mathbb{N}$ such that $j + k - 1 \geq i_0$, it holds $|y_{j,k} - a| \leq \varepsilon$. Therefore, $\mathbf{Y} \in c_2^{a,\mathrm{sum}}$ which completes the proof.

Theorem 3 ([11]) *The selection principle $\mathsf{S}_1^{(\mathsf{d})}(c^{0,P}_{2,+}, \ell^D_{2,\mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})})$ is satisfied.*

Proof Let $(\mathbf{X}^{k,l} : k,l \in \mathbb{N})$ be a double sequence of double sequences $\mathbf{X}^{k,l} = (x^{k,l}_{m,n})_{m,n\in\mathbb{N}}$ in $c^{0,P}_{2,+}$. We construct the double sequence $\mathbf{Y} = (y_{k,l})_{k,l\in\mathbb{N}}$ in the following way ($n \in \mathbb{N}$):

[Step 1: $(k,l) = (1,1)$] Pick $y_{1,1}$ in the double sequence $\mathbf{X}^{1,1}$ such that $y_{1,1} \leq 1$;

[Step 2: $(k,l) \in \{(1,2),(2,1),(2,2)\}$] Choose $y_{k,l} \in \mathbf{X}^{k,l}$ so that $y_{k,l} \leq \frac{1}{2^2} \cdot \frac{v_1(\mathbf{Y})}{3}$;

[Step n: $(k,l) \in \{(i,n),(n,i) : i \leq n\}$] Choose $y_{k,l} \in \mathbf{X}^{k,l}$ so that $y_{k,l} \leq \frac{1}{n^2} \cdot \frac{v_{n-1}(\mathbf{Y})}{2n-1}$. And so on.

In this way we obtain that $\mathbf{Y}$ is a double sequence of positive real numbers.

Claim 1. $\mathbf{Y} \in \ell_2^D$.

Observe that for every $n \in \mathbb{N}$, $d_n(\mathbf{Y}) \leq \sum_{i=1}^n \frac{1}{i^2}$, since $d_n(\mathbf{Y}) = \sum_{i=1}^n v_i(\mathbf{Y})$. Therefore, the sequence $(d_n(\mathbf{Y}))_{n\in\mathbb{N}}$ converges, i.e. $Y \in \ell_2^D$.

Claim 2. $\omega(\mathbf{S}(\mathbf{Y})) = (\omega_n(\mathbf{S}(\mathbf{Y})))_{n\in\mathbb{N}} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})$.

First, notice that for each $n \in \mathbb{N}$, $\omega_n(\mathbf{S}(\mathbf{Y})) = D - \Sigma\mathbf{Y} - d_n(\mathbf{Y})$. For sufficiently large $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{\omega_{n+1}(\mathbf{S}(\mathbf{Y}))}{\omega_n(\mathbf{S}(\mathbf{Y}))} &= \frac{D\text{–}\Sigma\mathbf{Y} - d_{n+1}(\mathbf{Y})}{D\text{–}\Sigma\mathbf{Y} - d_n(\mathbf{Y})} = 1 - \frac{d_{n+1}(\mathbf{Y}) - d_n(\mathbf{Y})}{D\text{–}\Sigma\mathbf{Y} - d_n(\mathbf{Y})} \\
&= 1 - \frac{v_{n+1}(\mathbf{Y})}{v_{n+1}(\mathbf{Y}) + v_{n+2}(\mathbf{Y}) + \ldots} = 1 - \frac{1}{1 + \frac{v_{n+2}(\mathbf{Y})}{v_{n+1}(\mathbf{Y})} + \frac{v_{n+3}(\mathbf{Y})}{v_{n+1}(\mathbf{Y}) + \ldots}} \\
&= 1 - \frac{1}{1 + \frac{v_{n+2}(\mathbf{Y})}{v_{n+1}(\mathbf{Y})} + \frac{v_{n+3}(\mathbf{Y})}{v_{n+2}(\mathbf{Y})} \cdot \frac{v_{n+2}(\mathbf{Y})}{v_{n+1}(\mathbf{Y})} + \ldots} \\
&\leq 1 - \frac{1}{1 + \frac{v_{n+2}(\mathbf{Y})}{v_{n+1}(\mathbf{Y})} + \frac{v_{n+3}(\mathbf{Y})}{v_{n+2}(\mathbf{Y})} + \ldots}.
\end{aligned}$$

Since the series $\sum_{n=1}^\infty \frac{v_{n+1}(\mathbf{Y})}{v_n(\mathbf{Y})}$ is convergent, we have that $\frac{v_{n+2}(\mathbf{Y})}{v_{n+1}(\mathbf{Y})} + \frac{v_{n+3}(\mathbf{Y})}{v_{n+2}(\mathbf{Y})} + \ldots$ tends to 0 for $n \to \infty$. Thus we conclude

$$\lim_{n\to\infty} \frac{\omega_{n+1}(\mathbf{S}(\mathbf{Y}))}{\omega_n(\mathbf{S}(\mathbf{Y}))} = 0$$

which means that $\omega(\mathbf{S}(\mathbf{Y})) \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})$, i.e. $\mathbf{Y} \in \ell^D_{2,\mathsf{Tr}(\mathsf{R}_{\mathsf{s},-\infty})}$.

This completes the proof of the theorem.

Remark 1 The double sequence $\mathbf{Y}$ in the proof of the previous theorem P-converges to 0, i.e. $\mathbf{Y} \in c_{2,+}^{0,P}$. Indeed, since $\mathbf{Y} \in \ell_2^D$, for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $v_n(\mathbf{Y}) \leq \varepsilon$ for all $n \geq n_0$. It follows that for all $p, q \in \mathbb{N}$, $y_{n_0+p,n_0+q} \leq \varepsilon$, hence $\mathbf{Y} \in c_{2,+}^{0,P}$.

Below, we give another result involving translational rapid variability.

Because the selection property S_1 is monotone in the second coordinate, by Proposition 2, the following theorem is an improvement of Corollary 1 (and Theorem 1).

Theorem 4 ([2]) *The selection principle* $\mathsf{S}_1^{(d)}(c_{2,+}^{0,P}, \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty}))$ *is satisfied.*

Proof 1 *Let* $(\mathbf{X}^{j,k} = (x_{m,n}^{j,k})_{m,n\in\mathbb{N}} : j,k \in \mathbb{N})$ *be a double sequence of double sequences in* $c_{2,+}^{0,P}$. *We construct a new double sequence* $\mathbf{Y} = (y_{j,k})_{j,k\in\mathbb{N}}$ *as follows.*

1. $y_{1,1} = x_{m,n}^{1,1}$ *for arbitrary (and fixed)* $m, n \in \mathbb{N}$.
2. *Let* $i \geq 2$.

(i) Choose $y_{i,1} = x^{i,1}_{m,n} \in \mathbf{X}^{i,1}$ *so that* $y_{i,1} < \left(\frac{1}{2}\right)^i y_{i-1,1}$. *For* $p \in \{2,3,\cdots,i-1\}$ *pick* $y_{i,p} = x^{i.p}_{m,n}$ *such that* $y_{i,p} < \left(\frac{1}{2}\right)^i y_{i,p-1}$ *and* $y_{i,p} < \left(\frac{1}{2}\right)^i y_{i-1,p}$.

(ii) Similarly, $y_{1,i} = x^{1,i}_{m,n} \in \mathbf{X}^{1,i}$ *such that* $y_{1,i} < \left(\frac{1}{2}\right)^i y_{1,i-1}$. *Select also* $y_{p,i} = x^{p,i}_{m,n}$ *such that* $y_{p,i} < \left(\frac{1}{2}\right)^i y_{p-1,i}$ *and* $y_{p.i} < \left(\frac{1}{2}\right)^i y_{p,i-1}$.

(iii) Finally, choose $y_{i,i}$ *to be some* $x^{i,i}_{m,n} \in \mathbf{X}^{i,i}$ *such that* $y_{i,i} < \left(\frac{1}{2}\right)^i \min\{y_{i,i-1}, y_{i-1,i}\}$.

It remains to prove that $\mathbf{Y} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$. *Let* $\varepsilon > 0$ *and* $\alpha, \beta \geq 0$ *with* $\max\{\alpha,\beta\} \geq 1$ *be given. Denote* $h = h(\alpha,\beta) = [\alpha] + [\beta]$. *Choose* $r_0 \in \mathbb{N}$ *such that* $(\frac{1}{2})^r \leq \varepsilon$ *for all* $r \geq r_0$. *For* $j \geq r_0$, $k \geq r_0$ *we have*

$$\frac{y_{j+1,k}}{y_{j,k}} \leq \left(\frac{1}{2}\right)^{r_0+1} \quad \textit{and} \quad \frac{y_{j,k+1}}{y_{j,k}} \leq \left(\frac{1}{2}\right)^{r_0+1},$$

and thus

$$\frac{y_{[j+\alpha],[k+\beta]}}{y_{j,k}} = \frac{y_{j+[\alpha],k+[\beta]}}{y_{j,k}} \leq \left(\frac{1}{2}\right)^{(r_0+1)h} \leq \left(\frac{1}{2}\right)^{r_0} \leq \varepsilon.$$

This means that $\mathbf{Y} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$.

Theorem 5 ([8]) *Let* $a \in \mathbb{R}$ *and let* $\leq_\varphi$ *be as above. Then the selection hypothesis* $\mathsf{S}^\varphi_1(c^{a,P}_2, \overline{c}^{a,P}_2)$ *is satisfied.*

Proof Let $(\mathbf{X}^k : k \in \mathbb{N})$, $\mathbf{X}^k = (x^k_{m,n})_{m,n\in\mathbb{N}}$, be a sequence in $c^{a,P}_2$. Construct a double sequence $\mathbf{Y} = (y_{s,t})_{s,t\in\mathbb{N}}$ as follows.

Fix $k \in \mathbb{N}$. Let $(s(k), t(k)) = \varphi^{-1}(k)$, and let $p(k) = \max\{s(k), t(k)\}$. There is $n_0(k) \in \mathbb{N}$ such that $|x^k_{m,n} - a| < 2^{-p(k)}$ for all $m, n \geq n_0(k)$. Set $y_{s(k),t(k)} = x^k_{n_0(k),n_0(k)}$ and $Y = (y_{s(k),t(k)})_{k\in\mathbb{N}}$. Then, by the construction, $\mathbf{Y} \in \overline{c}^a_2$ and $\mathbf{Y}$ has exactly one common element with $\mathbf{X}^k$ for each $k \in \mathbb{N}$, i.e. $\mathbf{Y}$ is the desired selector.

Theorem 6 ([2]) *The selection principle* $\mathsf{S}^\varphi_1(c^{0,P}_{2,+}, \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty}))$ *is satisfied.*

1.3 α_2 selection principle and double sequences

Now we give certain results about applications of α_2-type selection principles to double sequences.

Lemma 1 [8] *For* $a \in \mathbb{R}$, *the selection principle* $\alpha_2(c^{a,P}_2, \overline{c}^{a,P}_2)$ *is satisfied.*

Proof Let $(\mathbf{X}^k : k \in \mathbb{N})$ be a sequence of elements from $c_2^{a,P}$ and let for each $k \in \mathbb{N}$, $\mathbf{X}^k = (x^k_{m,n})_{m,n\in\mathbb{N}}$.

1. Form first an increasing sequence $j_1 < j_2 < \cdots < j_i < \cdots$ in $\mathbb{N}$ so that:

1.a. $j_1 = \min\{n_0 \in \mathbb{N} : \left|x^1_{m,n} - a\right| \leq \frac{1}{2}\ \forall m \geq n_0 \text{ and } \forall n \geq n_0\}$;

1.b. Let $i \geq 2$. Find $p_i = \min\{n_0 \in \mathbb{N} : \left|x^i_{m,n} - a\right| \leq \frac{1}{2^i}\ \forall m, n \geq n_0\}$, and then define

$$j_i = p_i,$$

if $\mathrm{p}_i > j_{i-1}$;$\mathrm{j}_{i-1} + 1$, if $\mathrm{p}_i \leq j_{i-1}$.

2. Define now a double sequence $\mathbf{Y} = (y_{s,t})_{s,t\in\mathbb{N}}$ in this way:

2.a. $y_{s,t} = x^1_{s,t}$ for each $1 \leq s < j_2$, $t \in \mathbb{N}$, and each $1 \leq t < j_2$, $s \in \mathbb{N}$;

2.b. For $i \geq 2$, $y_{s,t} = x^i_{s,t}$, for $j_i \leq s < j_{i+1}$, $t \geq j_i$, and $j_i \leq t < j_{i+1}$, $s \geq j_i$.

By construction, $\mathbf{Y} \in c_2^a$ and $\mathbf{Y}$ has infinitely many common elements with each $\mathbf{X}^k$, $k \in \mathbb{N}$, i.e. the selection principle $\alpha_2(c_2^a, c_2^a)$ is satisfied.

Remark 2 Using the technique from [4] we can prove that the double sequence $\mathbf{Y}$ in the proof of the previous lemma can be chosen in such a way that Y has infinitely many common elements with each $\mathbf{X}^k$, $k \in \mathbb{N}$, but on the same (corresponding) positions.

Let for each $k \in \mathbb{N}$, $\mathbf{x}^k$ denote the sequence $(x^k_{m,m})_{m\in\mathbb{N}}$. Then each $\mathbf{x}^k$ converges to a, so that we have the sequence $(\mathbf{x}^k : k \in \mathbb{N})$ of sequences converging to a. Let $2 = p_1 < p_2 < p_3 < \cdots$ be a sequence of prime natural numbers. Take sequence $\mathbf{x}^1 = (x^1_{m,m})_{m\in\mathbb{N}}$. For each $i \in \mathbb{N}$, replace the elements of $\mathbf{x}^1$ on the positions p_i^h, $h \in \mathbb{N}$, by the corresponding elements of the sequence $\mathbf{x}^{i+1}$. One obtains the sequence $(z_m)_{m\in\mathbb{N}}$ converging to a which has infinitely many common elements with each $\mathbf{x}^k$ on the same positions as in $\mathbf{x}^k$. Define now the double sequence $\mathbf{Y} = (y_{s,t})_{s,t\in\mathbb{N}}$ so that $y_{s,s} = z_s$, $s \in \mathbb{N}$, and $y_{s,t} = a$ whenever $s \neq t$. By construction, $\mathbf{Y} \in \overline{c}_2^{a,P}$ and has infinitely many common positions with each $\mathbf{X}^k$.

Theorem 7 *Let $a \in \mathbb{R}$ be given. The selection principle $\alpha_2^{(d)}(c_2^{a,P}, \overline{c}_2^{a,P})$ is true.*

Proof Let $(\mathbf{X}^{j,k} : j, k \in \mathbb{N})$ be a double sequence of elements in $c_2^{a,P}$ and let $\mathbf{X}^{j,k} = (x^{j,k}_{m,n})_{m,n\in\mathbb{N}}$. In a standard way form from this double sequence a sequence can be arranged $(x^i_{m,n})_{m,n\in\mathbb{N}}$. Apply now Lemma 1 to this sequence and find a double sequence $\mathbf{Y} \in c_2^{a,P}$ such that $\mathbf{Y} \cap \mathbf{X}^i$ is infinite for each $i \in \mathbb{N}$. But then $\mathbf{Y} \cap \mathbf{X}^{j,k}$ is infinite for all $j, k \in \mathbb{N}$.

Remark 3 Notice that the double sequence $\mathbf{Y}$ from the proofs of Lemma 1 and Theorem 7 satisfies: (a) Y is bounded; (b) Y is regular and $\lim_{n\to\infty} y_{m,n} = \lim_{m\to\infty} y_{m,n} = a$ for each $m \in \mathbb{N}$ and each $n \in \mathbb{N}$.

From Theorem 7 we have the following corollary.

Corollary 2 *Let $a \in \mathbb{R}$ be given. The selection principle $\alpha_2^{(d)}(c_2^{a,P}, c_2^{a,P})$ is true.*

This corollary can be improved by replacing the second coordinate in it with a smaller class.

Theorem 8 ([2]) *The selection principle $\alpha_2^{(d)}(c_{2,+}^{0,P}, \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty}))$ is satisfied.*

Proof Let $(\mathbf{X}^{j,k} = (x_{m,n}^{j,k})_{m,n\in\mathbb{N}} : j,k \in \mathbb{N})$ be a double sequence of double sequences belonging to $c_{2,+}^{0,P}$. We are going to create a new double sequence $\mathbf{Y} = (y_{p,q})_{p,q\in\mathbb{N}}$ in the following way.

Step 1. Using some standard method arrange the given double sequence $(\mathbf{X}^{j,k} : j,k \in \mathbf{N})$ of double sequences in a sequence $(\mathbf{x}^r = (x_{n,m}^r)_{m,n\in\mathbb{N}} : r \in \mathbb{N})$ from $c_{2,+}^{0,P}$.

Step 2. Consider the sequence of sequences $(x_{n,n}^r)_{n,r\in\mathbb{N}}$. Notice that for each $r \in \mathbb{N}$, $(x_{n,n}^r)_{n,r\in\mathbb{N}} \in \mathbb{S}_0$, where $\mathbb{S}_0$ denotes the set of all sequences of positive real numbers converging to 0 (see, for example, [7]). Let $\mathbb{T}_0$ be the set $\{(a_n)_{n\in\mathbb{N}} : a_1 > 0, a_{n+1} \leq \frac{a_n}{n+1}\}$ of sequences of positive real numbers. It holds $\mathbb{T}_0 \subsetneqq \mathbb{S}_0$ and the selection principle $\mathsf{S}_1(\mathbb{S}_0, \mathbb{T}_0)$ is satisfied.

Step 3. (In this part of the proof we use some techniques from [4]) Take an increasing sequence $2 = p_1 < p_2 < p_3 < \dots$ of prime numbers and a fixed $r \in \mathbb{N}$. Consider subsequences $(x_{p_t^n, p_t^n}^r)_{t\in\mathbb{N}}$, of the sequence $(x_{n,n}^r)_{n\in\mathbb{N}}$. These subsequences are in the class $\mathbb{S}_0$. Varying t and r in $\mathbb{N}$, arrange those subsequences in a sequence of sequences from $\mathbb{S}_0$. Apply $\mathsf{S}_1(\mathbb{S}_0, \mathbb{T}_0)$ and find a sequence $\mathbf{z} = (z_q)_{q\in\mathbb{N}} \in \mathbb{T}_0$ such that $\mathbf{z}$ has infinitely many elements with the sequence $(x_{n,n}^r)_{n\in\mathbb{N}}$ for each $r \in \mathbb{N}$. In other words, we conclude that the selection principle $\alpha_2(\mathbb{S}_0, \mathbb{T}_0)$ is true.

Let now $y_{q,q} = z_q$, $q \in \mathbb{N}$. For $q \geq 2$ we choose $y_{u,q} = \sqrt{u+1} \cdot y_{u+1,q}$ for $u \in \{1, 2, \cdots, q-1\}$, and $y_{q,u} = \sqrt{u+1} \cdot y_{q,u+1}$. It is easy to see that the double sequence $\mathbf{Y}$ constructed in this way has infinitely many common elements with each double sequence $\mathbf{X}^{j,k} = (x_{m,n}^{j,k})_{m,n\in\mathbb{N}}$ for arbitrary and fixed $(j,k) \in \mathbb{N} \times \mathbb{N}$.

It remains to prove $\mathbf{Y} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2 -\infty})$. Let $\varepsilon > 0$ and $\alpha \geq 0$, $\beta \geq 0$ with $\max\{\alpha, \beta\} \geq 1$, be given. Set $h = [\alpha] + [\beta]$. There is $n_0 \in \mathbb{N}$ such that $\left(\frac{1}{\sqrt{N+1}}\right)^h \leq \varepsilon$ for each $N \in \mathbb{N}$ with $N \geq n_0$ ($n_0 \geq \varepsilon^{-(2/h)} - 1$). For $p, t \geq n_0$ we have

$$\frac{y_{p+1,t}}{y_{p,t}} \leq \frac{1}{\sqrt{n_0+1}} \quad \text{and} \quad \frac{y_{p,t+1}}{y_{p,t}} \leq \frac{1}{\sqrt{n_0+1}}.$$

So
$$\frac{y_{[p+\alpha],[t+\beta]}}{y_{p,t}} = \frac{y_{p+[\alpha],t+[\beta]}}{y_{p,t}} \left(\frac{1}{\sqrt{n_0+1}}\right)^h \le \varepsilon,$$
i.e. $\mathbf{Y} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s}_2,-\infty})$.

Theorem 9 ([11]) *The selection principle $\alpha_2^{(d)}(c_{2,+}^{0,P}, \ell_2^D)$ is satisfied.*

Proof Let $(\mathbf{X}^{k,l} : k,l \in \mathbb{N})$ be a double sequence of double sequences $\mathbf{X}^{k,l} = (x_{m,n}^{k,l})_{m,n\in\mathbb{N}}$ in $c_{2,+}^{0,P}$. In a standard way we reorganize $(\mathbf{X}^{k,l} : k,l \in \mathbb{N})$ in a sequence $(\mathbf{X}^t = (x_{m,n}^t)_{m,n\in\mathbb{N}} : t \in \mathbb{N})$ of double sequences such that for each $t \in \mathbb{N}$, the double sequence $\mathbf{X}^t \in c_{2,+}^{0,P}$. We construct the double sequence $\mathbf{Y} = (y_{i,j})_{i,j\in\mathbb{N}}$ as follows.

[Step 1: $i = 1$]: Let $\mathbf{Y}_1 = (y_{s,t}^1)_{s,t\in\mathbb{N}}$ be a double sequence such that for all $s,t \in \mathbb{N}$ it holds $0 < y_{s,t}^1 \le \frac{1}{M^2(2M-1)}$, where $M = \max\{s,t\}$.

[Step 2: $i \ge 2$] Suppose that double sequences $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i-1}$ be defined. We construct the sequence $\mathbf{Y}_i$. Take an increasing sequence $(p_i)_{i\in\mathbb{N}}$ of prime numbers with $p_1 = 2$ and a bijection $\varphi_i : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{q=1}^{\infty} x^i_{p_i^{\varphi_i(q)}}$ converges and $\le \frac{1}{i^2}$. We replace now elements $y_{p_i^{\varphi_i(q)},p_i^{\varphi_i(q)}} \in \mathbf{Y}_{i-1}$ with elements $x^i_{p_i^{\varphi_i(q)},p_i^{\varphi_i(q)}} \in \mathbf{X}^i$. Proceed with this procedure as $i \to \infty$. We obtain the double sequence $\mathbf{Y}$ as required. Indeed, evidently $\mathbf{Y} \in c_{2,+}^{0,P}$ and $\mathbf{Y} \in \ell_2^D$, and, by construction, $\mathbf{Y}$ has infinitely many common elements with each sequence $\mathbf{X}^{k,l}$.

Theorem 10 ([9]) *For a given $a \in \mathbb{R}$ the selection principle $\alpha_2^{(d)}(c_2^{a,P}, c_2^{a,\mathrm{sum}})$ holds.*

Proof Let $(\mathbf{X}^{j,k} = (x_{m,n}^{j,k})_{m,n\in\mathbb{N}} : j,k \in \mathbb{N})$ be a double sequence of double sequences in $c_{2,+}^{0,P}$. We are going to form a new double sequence $\mathbf{Y} = (y_{j,k})_{j,k\in\mathbb{N}}$ as follows.

For $j,k \in \mathbb{N}$ with $j \ne k$ we take $y_{j,k} = a$. Let $j = k$. Take first an increasing sequence $(p_i)_{i\in\mathbb{N}}$, $2 = p_1 < p_2 < p_3 < \ldots$, of prime numbers. Then, the initial double sequence of double sequences organize as a sequence $\mathbf{x} = (x_{m,n}^i)_{i\in\mathbb{N}}$. For $i \ge 2$ we take $y_{j,j} = x^i_{p_i^s,p_i^s}$ if $j = p_i^s$ for some $s \in \mathbb{N}$. If $\left|x^i_{p_i^s,p_i^s} - a\right| > \frac{1}{2^i}$, then already defined $y_{j,j}$ replace with $y_{j,j} = a$. Put $j_i = \min\{j \in \mathbb{N} : y_{j,j} = x^i_{p_i^s,p_i^s}\}$. If $j_{i+1} < j_i$, then already defined $y_{j,j}$, $j \in \{j_{i+1}, j_{i+1}+1, \ldots, j_i - 1\}$, replace by putting $y_{j,j} = a$. If p_i and s with $j = p_i^s$ do not exist, then take $y_{j,j} = a$. According to the construction of $\mathbf{Y}$ we have:

(1) $\mathbf{Y} = (y_{j,k})_{j,k\in\mathbb{N}} \in c_2^{a,\mathrm{sum}}$, and

(2) $\mathbf{Y}$ has infinitely many common elements (at the same positions) with every double sequence $\mathbf{X}^{j,k}$.

This completes the proof.

Double sequences of positive real numbers $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ and $\mathbf{Y} =$

$(y_{m,n})_{m,n\in\mathbb{N}}$ are said to be *P-strongly asymptotically equivalent* (or *P-asymptotically equal*), denoted $\mathbf{X} \overset{\mathrm{P}}{\sim} \mathbf{Y}$, if for every $\varepsilon > 1$ there is $n_0 = n_0(\varepsilon)$ such that $\frac{1}{\varepsilon} \leq \frac{x_{m,n}}{y_{m,n}} \leq \varepsilon$ for all $m \geq n_0$, $n \geq n_0$. The relation $\overset{P}{\sim}$ is an equivalence relation od the set of double sequences of positive real numbers, and $[\mathbf{X}]_{\mathrm{P}}$ denotes the equivalence class of $\mathbf{X}$.

In a similar way we define *max-strong asymptotic equivalence* relation $\overset{\max}{\sim}$ by

$$\mathbf{X} \overset{\max}{\sim} \mathbf{Y} \Leftrightarrow \max{-}\lim \frac{x_{m,n}}{y_{m,n}} = 1$$

and *sum-strong asymptotic equivalence* $\overset{\text{sum}}{\sim}$ by

$$\mathbf{X} \overset{\text{sum}}{\sim} \mathbf{Y} \Leftrightarrow \text{sum}{-}\lim \frac{x_{m,n}}{y_{m,n}} = 1$$

The corresponding equivalence classes of $\mathbf{X}$ are denoted by $[\mathbf{X}]_{\max}$ and $[\mathbf{X}]_{\text{sum}}$, respectively. Evidently,

$$[\mathbf{X}]_{\text{sum}} \subsetneq [\mathbf{X}]_{\max} \subsetneq [\mathbf{X}]_{\mathrm{P}}.$$

Theorem 11 *Let $\mathbf{X} = (x_{m,n})_{m,n\in\mathbf{N}}$ be a given double sequence of positive real numbers. Then the selection principle $\alpha_2^{(d)}([\mathbf{X}]_{\mathrm{P}}, [\mathbf{X}]_{\text{sum}})$ holds.*

Proof Let $(\mathbf{Y}^{j,k} = (y^{j,k}_{m,n})_{m,n\in\mathbb{N}} : j,k \in \mathbb{N})$ be a double sequence of double sequences in $[\mathbf{X}]_{\mathrm{P}}$. Take an increasing sequence $(p_i)_{i\in\mathbb{N}}$ of prime numbers with $p_1 = 2$, and arrange the double sequence $(\mathbf{Y}^{j,k} : j,k \in \mathbb{N})$ into a sequence $(\mathbf{Y}^i = (y^i_{m,n})_{m,n\in\mathbb{N}} : i \in \mathbb{N})$ of double sequences. Then we define the double sequence $\mathbf{Z} = (z_{j,k})_{j,k\in\mathbb{N}}$ as follows.

Inductively we construct a sequence $(\mathbf{Z}_i : i \in \mathbb{N})$ of double sequences. Let $\mathbf{Z}_1 = \mathbf{X}$. Then for $i \geq 2$ we construct the double sequence $\mathbf{Z}_i$ by replacing elements at positions (p_i^s, p_i^s), $s \in \mathbb{N}$, in the double sequence $\mathbf{Z}_{i-1}$ by elements $y^i_{p_i^s,p_i^s}$, $s \in \mathbb{N}$, if

$$\frac{2^i - 1}{2^i} \leq \frac{y^i_{p_i^s,p_i^s}}{x_{p_i^s,p_i^s}} \leq \frac{2^i}{2^i - 1}.$$

Continuing this procedure as $i \to \infty$ we get the required double sequence $\mathbf{Z}$. Indeed, by construction, Z has infinitely many common elements (at the same positions) with double sequences $\mathbf{Y}^i$, $i \in \mathbb{N}$, and $\mathbf{Z} \in [\mathbf{X}]_{\text{sum}}$.

The following result is given without proof.

Theorem 12 *Let $a \in \mathbb{R}$ be given. Then the selection principle $\mathsf{S}_1^{\varphi}(c_2^{a,P}, c_2^{a,\text{sum}})$ holds.*

Theorem 13 *Let $a \in \mathbb{R}$ and let $(\mathbf{X}^k : k \in \mathbb{N})$ be a sequence of double sequences in $c_2^{a,P}$, $\mathbf{X}^k = (x^k_{m,n})_{m,n\in\mathbb{N}}$. Then there is a double sequence $\mathbf{Y} = (y_{s,t})_{s,t\in\mathbb{N}}$ in $c_2^{a,P}$ such that for each $k \in \mathbb{N}$ the set $\{(s,t) \in \mathbb{N}\times\mathbb{N} : y_{s,t} = x^k_{m,n}$ for some $(m,n) \in \mathbb{N}\times\mathbb{N}\}$ is infinite.*

Proof The double sequence $\mathbf{Y}$ is defined in the following way:

Let $k \in \mathbb{N}$. There is $i_k \in \mathbb{N}$ such that $\left|x_{m,n}^k - a\right| < 2^{-k}$ for all $m, n \geq i_k$. Let

$$s^* = i_k,$$

for s=k, $i_k + p$, for s=k+p, p$\in \mathbb{N}$,

and

$$t^* = i_k,$$

for t=k, $i_k + p$, for t=k+p, p$\in \mathbb{N}$.

For $t \geq k$ let $y_{k,t} = x_{i_k,t^*}^k$, and for $s \geq k$ let $y_{s,k} = x_{s^*,i_k}^k$. The double sequence $\mathbf{Y} = (y_{s,t})_{s,t\in\mathbb{N}}$ constructed in this way is as required, because Y has the following properties:

(1) $\mathbf{Y} \in c_2^{a,P}$;

(2) The set $B^k = \{y_{k,t} : t \geq k\} \cup \{y_{s,k} : s \geq k\}$ is a subset of $A^k = \{x_{m,n}^k : m, n \in \mathbb{N}\}$;

(3) For each $k \in \mathbb{N}$, B^k is countable;

(4) $\bigcup_{k\in\mathbb{N}} B^k = \{y_{s,t} : s, t \in \mathbb{N}\}$.

Another similar result is given in the next theorem.

Theorem 14 *Let $a \in \mathbb{R}$ and let $(\mathbf{X}^k : k \in \mathbb{N})$ be a sequence of double sequences in $c_2^{a,P}$, $\mathbf{X}^k = (x_{m,n}^k)_{m,n\in\mathbb{N}}$. Then there is a double sequence $\mathbf{Y} = (y_{s,t})_{s,t\in\mathbb{N}}$ in $c_2^{a,P}$ which has one common row with $\mathbf{X}^k$ for each $k \in \mathbb{N}$.*

Proof For each $k \in \mathbb{N}$ there is $n_0(k) \in \mathbb{N}$ such that $\left|x_{m,n}^k - a\right| < 2^{-k}$ for all $m, n \geq n_0(k)$, $n_0(k_1) > n_0(k_2)$ whenever $k_1 > k_2$, and $n_0(k) \geq \min\left\{i(k) \in \mathbb{N} : \left|x_{m,n}^{k+1} - a\right| < 2^{-k} \text{ for all } m, n \geq i(k)\right\}$. Then the desired double sequence $\mathbf{Y}$ is defined in such a way that its $n_0(k)$th row is the $n_0(k)$th row of $\mathbf{X}^k$, i.e. $y_{n_0(k),n} = x_{n_0(k),n}^k$ $(n \in \mathbb{N})$, and $y_{s,t} = a$ otherwise. Let us prove that $\mathbf{Y} \in c_2^{a,P}$. Indeed, if $\varepsilon > 0$ is given, then choose $p \in \mathbb{N}$ such that $2^{-p} < \varepsilon$. Then for each $k \in \mathbb{N}$ we have $\left|x_{m,n}^k - a\right| < \varepsilon$ for all $m, n \geq p$. By construction of $\mathbf{Y}$ we have actually that $|y_{m,n} - a| < \varepsilon$ for all $m, n \geq p$, i.e. $\mathbf{Y} \in c_2^{a,\mathrm{P}}$.

1.4 Double sequences and the exponent of convergence

The notion of exponent of convergence of single real sequences play an important role in the theory of convergence/divergence of sequences. This notion was implicitly defined by Pringsheim [28]. In 1931, Serbian mathematician M. Petrović [26] introduced the notion of sequence of exponents of convergence and gave an important contribution to this field.

The first application of the exponent of convergence for single sequences in the theory of selection was presented in [6]. The authors of [3] defined the exponent of convergence for double sequences and studied its applications in selection principles theory.

Definition 2 ([3] A real number λ is said to be the *exponent of convergence* (in the Prinsheim sense) of a double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}} \in \mathsf{c}_{2,+}^{0,\mathrm{P}}$ if for every $\varepsilon > 0$, the double sequence $\mathbf{X}_\varepsilon^+ = (x_{m,n}^{\lambda+\varepsilon})_{m,n\in\mathbb{N}}$ has a finite P-sum, while the double sequence $\mathbf{X}_\varepsilon^- = (x_{m,n}^{\lambda-\varepsilon})_{m,n\in\mathbb{N}}$ does not have.

If for every $\varepsilon > 0$, the double sequence $\mathbf{X}_\varepsilon = (x_{m,n}^{\varepsilon})_{m,n\in\mathbb{N}}$ does not have a finite P-sum, then we say that $\lambda = \infty$ is the exponent of convergence of $\mathbf{X}$.

Let $\lambda \in [0,\infty]$ and let $\mathsf{c}_{2,+}^{0,\mathrm{P}}(\lambda)$ denote the set of all double sequences from $c_{2,+}^{0,\mathrm{P}}$ which P-converge to zero and whose exponent of convergence is λ.

Theorem 15 ([3]) *The selection principle* $\mathsf{S}_1^{(\mathsf{d})}(\mathsf{c}_{2,+}^{0,\mathrm{P}}, \mathsf{c}_{2,+}^{0,\mathrm{P}}(\lambda))$ *is satisfied for* $\lambda = 0$.

Proof Let $(\mathbf{X}^{k,l} = (x_{m,n}^{k,l})_{m,n\in\mathbb{N}} : k,l \in \mathbb{N})$ be a double sequence of double sequences from the class $\mathsf{c}_{2,+}^{0,\mathrm{P}}$. We will form a double sequence $\mathbf{Y} = (y_{k,l})_{k,l\in\mathbb{N}}$ in the following way:

[Step 1: $n = 1$] Choose an element $y_{1,1}$ from double sequence $\mathbf{X}^{1,1}$ such that $y_{1,1} \leq \frac{1}{2}$.

[Step 2: $n \geq 2$] For $(k,l) \in \{(i,n),(n,i) : i \leq n\}$, choose $y_{k,l}$ from the double sequence $\mathbf{X}^{(k,l)}$ such that $y_{k,l} \leq \frac{1}{2^n}$.

Claim 1. $\mathbf{Y} \in \mathsf{c}_{2,+}^{0,\mathrm{P}}$.

For any $n \in \mathbb{N}$ we have

$$v_n(\mathbf{Y}) \leq \frac{2n-1}{2^n}$$

and thus

$$0 < \sum_{n=1}^{\infty} v_n(\mathbf{Y}) \leq \sum_{n=1}^{\infty} \frac{2n-1}{2^n}.$$

The series $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$ is convergent in $\mathbb{R}$, hence the series $\sum_{n=1}^{\infty} v_n(y)$ is convergent in $\mathbb{R}$. Thus, we conclude that the double sequence $\mathbf{Y}$ has the finite diagonal sum

$$D\text{–}\Sigma\mathbf{Y} = \lim_{n\to\infty} d_n(\mathbf{Y}) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \sum_{l=1}^{n} y_{k,l} \right)$$

in $\mathbb{R}$ (more about diagonal sums can be seen in [11]). By results obtained in [11, Proposition 1.2], we conclude that $\mathbf{Y} \in \mathsf{c}^{0,\mathsf{P}}_{2,+}$.

Claim 2. For any $\varepsilon > 0$, $\mathbf{Y}^{+}_{\varepsilon} = (x^{0+\varepsilon}_{m,n})_{m,n\in\mathbb{N}}$ has finite P-sum.

For $n, k, l \in \mathbb{N}$ we have $y^{\varepsilon}_{k,l} \leq \frac{1}{2^{\varepsilon n}}$. Therefore, for the double sequence $\mathbf{Y}^{\varepsilon} = (y^{\varepsilon}_{k,l})_{k,l\in\mathbb{N}}$ it holds $v_n(\mathbf{Y}^{\varepsilon}) \leq \frac{2n-1}{2^{\varepsilon n}}$, and thus

$$D\text{–}\Sigma\mathbf{Y}^{\varepsilon} \leq \sum_{n=1}^{\infty} \frac{2n-1}{2^{\varepsilon n}}.$$

Since

$$\lim_{n\to\infty} \frac{(2n+1)2^{\varepsilon n}}{(2n-1)2^{\varepsilon(n+1)}} = \frac{1}{2^{\varepsilon}} < 1,$$

the series $\sum_{n=1}^{\infty} \frac{2n-1}{2^{\varepsilon n}}$ is convergent. Again by results from [11, Proposition 1.2] we obtain that the double sequence Y^{ε} has finite P-sum $P\text{–}\Sigma\mathbf{Y}^{+}_{\varepsilon}$.

Claim 3. For any $\varepsilon > 0$, $\mathbf{Y}^{-}_{\varepsilon} = (x^{0-\varepsilon}_{m,n})_{m,n\in\mathbb{N}|}$ does not have finite P-sum.

By construction of the double sequence $\mathbf{Y}$, we have that $\lim_{n\to\infty} y_{n,n} = 0$, which implies $\lim_{n\to\infty} y^{0-\varepsilon}_{n,n} = \infty$, and thus the double sequence $\mathbf{Y}$ does not P-converge to zero. By results from [11, Proposition 1.3] the double sequence $\mathbf{Y}^{-}_{\varepsilon}$ does not have finite P-sum.

The following theorem we give without proof.

Theorem 16 ([3]) *The selection principle* $\mathsf{S}^{\varphi}_1(\mathsf{c}^{0,\mathsf{P}}_{2,+}, \mathsf{c}^{0,\mathsf{P}}_{2,+}(\lambda))$ *is satisfied for* $\lambda = 0$.

Theorem 17 ([3]) *The selection principle* $\alpha^{(d)}_2(\mathsf{c}^{0,\mathsf{P}}_{2,+}, \mathsf{c}^{0,\mathsf{P}}_{2,+}(\lambda))$ *is satisfied for* $\lambda = 0$.

Proof Let $(\mathbf{X}^{k,l} = (x^{k,l}_{m,n})_{m,n\in\mathbb{N}} : k, l \in \mathbb{N})$ be a double sequence of double sequences from the class $\mathsf{c}^{0,\mathsf{P}}_{2,+}$. As in the proof of Theorem 15 create the double sequence $\mathbf{Y} = (y_{k,l})_{k,l\in\mathbb{N}}$; therefore, for a fixed $n \in \mathbb{N}$ we have that for $(k, l) \in \{(i, n), (n, i) : i \leq n\}$, $z_{k,l} \leq \frac{1}{2^n}$. By the standard method arrange given double sequence of double sequences in a sequence $(\mathbf{X}_t = (x^{k,l}_t)_{k,l\in\mathbb{N}} : t \in \mathbb{N})$ of double sequences belonging to $\mathsf{c}^{0,\mathsf{P}}_{2,+}$.

Take a sequence $2 = p_1 < p_2 < p_< \dots$ of prime numbers. For a fixed $t \in \mathbb{N}$, consider a sequence $(x^t_{p^s_t,p^s_t})_{s\in\mathbb{N}}$. Clearly, this sequence converges to zero when $s \to \infty$. There exist $s_{p_t} \in \mathbb{N}$ and a subsequence

$$(x^t_{p_t^{h(s)},p_t^{h(s)}}),$$

such that

$$\sum_{s=s_{p_t}}^{\infty} x^t_{p_t^{h(s)},p_t^{h(s)}} \leq \frac{1}{2^t}.$$

For each $s \geq s_{p_t}$ it holds $x^t_{p_t^{h(s)}, p_t^{h(s)}} \leq \frac{1}{2^t}$, which implies $x^{\varepsilon t}_{p_t^{h(s)}, p_t^{h(s)}} \leq \frac{1}{2^{\varepsilon t}}$ for the same s.

In the double sequence $\mathbf{Y}$, replace elements $y_{k,k}$, where $k = p_t^{h(s)}$ for $s \geq s_{p_t}$, with the elements $x^t_{p_t^{h(s)}, p_t^{h(s)}}$. This will be done for every $t \in \mathbb{N}$. In this way, we obtain the double sequence $\overline{\mathbf{Y}} = (\overline{y}_{k,l})$. Then we have

$$0 < D\text{–}\sum \overline{\mathbf{Y}} \leq D\text{–}\sum \mathbf{Y} + \sum_{t=1}^{\infty} \frac{1}{2^t} < \infty,$$

and therefore $P\text{-}\lim \overline{Y} = 0$. Moreover, the following hold:

(1) $\overline{y} \cap (x^t_{k,l})$ is an infinite set, for every $t \in \mathbb{N}$;

(2) $\overline{\mathbf{Y}} \in \mathsf{c}^{0,\mathsf{P}}_{2,+}$;

(3) $P\text{-}\lim \overline{\mathbf{Y}} \in \mathbb{R}$.

Let an arbitrary $\varepsilon > 0$ be given. Then

$$0 < D\text{–}\sum \overline{\mathbf{Y}}^{\varepsilon}_{+} \leq D\text{–}\sum \mathbf{Y}^{\varepsilon}_{+} + \sum_{t=1}^{\infty} \frac{1}{2^{\varepsilon t}} < \infty,$$

since $\lim_{t\to\infty} \frac{2^{\varepsilon t}}{2^{\varepsilon(t+1)}} = 2^{-\varepsilon} < 1$. Thus, $D\text{–}\sum \overline{\mathbf{Y}}^{\varepsilon}_{+}$ is finite, so the double sequence $\overline{\mathbf{Y}}^{\varepsilon}_{+}$ has a finite P-sum according to [11, Proposition 1.2].

Let now $\varepsilon < 0$ be arbitrary. Consider the double sequence $\overline{\mathbf{Y}}^{\varepsilon} = (\overline{y}^{\varepsilon})_{k,l}$. Since $\lim_{k\to\infty} \overline{y}_{k,k} = 0$ implies $\lim_{k\to\infty} \overline{y}^{\varepsilon}_{k,k} = \infty$, we can conclude that double sequence $\overline{\mathbf{Y}}^{\varepsilon}$ does not have finite P-sum.

This completes the proof of the theorem.

1.5 Concluding remarks

We have investigated here selection properties related mainly to P-limits of double sequences. There is another interesting notion in the theory of real double sequences. A number $L \in \mathbb{R}$ is said to be a *Pringsheim limit point* of a double sequence $\mathbf{X} = (x_{m,n})_{m,n\in\mathbb{N}}$ if there exist two increasing sequences $m_1 < m_2 \cdots < m_i, \ldots$ and $n_1 < n_2 \cdots < n_i, \ldots$ such that

$$\lim_{i\to\infty} x_{m_i,n_i} = L.$$

It would be worth to study selection properties related to the Pringsheim limit points instead of the P-limits. As far we know, there is no investigation in this direction so far.

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