

Article

Common Fixed Points Theorems for Self-Mappings in Menger PM-Spaces

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Abstract: The purpose of this paper is to prove that orbital continuity for a pair of self-mappings is a necessary and sufficient condition for the existence and uniqueness of a common fixed point for these mappings defined on Menger PM-spaces with a nonlinear contractive condition. The main results are obtained using the notion of R -weakly commutativity of type A_f (or type A_g). These results generalize some known results.

Keywords: probabilistic metric spaces; common fixed point; R -weakly commuting mappings; nonlinear contractive condition

MSC: 47H10; 54H25



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1. Introduction

One of the most important generalization of the Banach Contraction Mapping Principle [1] is the introduction of a nonlinear contractive principle by Boyd and Wong [2]. In 1971, Ćirić [3] introduced the notion of orbital continuity, as a generalization of continuity.

Definition 1 ([3]). *If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x \mid n = 0, 1, 2, \dots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_{i \rightarrow \infty} f^{m_i} x$ implies $fu = \lim_{i \rightarrow \infty} f f^{m_i} x$.*

Shastri et al. [4] defined the notion of orbital continuity for a pair of self-mappings.

Definition 2 ([4]). *If f and g are self-mappings of a metric space (X, d) and if $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a sequence in X such that $gx_n = fx_{n+1}$, $n = 0, 1, 2, \dots$, then the set $O(x_0, g, f) = \{gx_n \mid n = 0, 1, 2, \dots\}$ is called the (g, f) -orbit at x_0 and f (or g) is called (g, f) -orbitally continuous if $\lim_{n \rightarrow \infty} gx_n = u$ implies $\lim_{n \rightarrow \infty} f gx_n = fu$ (or $\lim_{n \rightarrow \infty} gx_n = u$ implies $\lim_{n \rightarrow \infty} gg x_n = gu$).*

The first common fixed theorem for a commutative pair of self-mappings was obtained by Jungck [5] as a generalization of the Banach contraction principle for classical metric spaces. There are many generalizations of commutativity for which were obtained common fixed theorems for a pair (or more) of mappings (see e.g., [6–12]). The notion of R -weak commutativity is introduced by Pant [13].

Definition 3 ([13]). *Two self-mappings f and g of a metric space (X, d) are called R -weakly commuting if there exists some real number $R > 0$, such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X . The mappings f and g are called point-wise R -weakly commuting on X if given x in X there exists $R > 0$, such that $d(fgx, gfx) \leq Rd(fx, gx)$.*

Using this notion, Pant [13] proved two common fixed point theorems for a pair of mappings, under the assumption that either one of the mappings is continuous. Patak et al. [14] improved these results by introducing the notion of R -weak commutativity of type A_g (or of type A_f).

Definition 4 ([14]). *Two self-mappings f and g of a metric space (X, d) are called R -weakly commuting of type A_g if there exists some real number $R > 0$, such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X . Similarly, the self-mappings f and g are called R -weakly commuting of type A_f if there exists some real number $R > 0$, such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X .*

Recently, Pant et al. [10] proved a common fixed point theorem of a metric space (X, d) for two self-mappings using the notion of R -weak commutativity of type A_f or of type A_g .

The first result from the fixed point theory in probabilistic metric spaces was obtained by Sehgal and Bharucha–Reid [15]. Since then, the fixed and common fixed point theorems for various contraction mappings in probabilistic metric spaces were investigated by many authors (see e.g., [16–23]).

In this paper, we prove that the orbital continuity for a pair of self-mappings is a necessary and sufficient condition for the existence and uniqueness of a common fixed point for these mappings if they are R -weakly commutings of type A_f or of type A_g with nonlinear contractive condition in the sense of Boyd and Wong, for Menger PM-spaces with arbitrary continuous t -norm. Topological methods for characterizing Menger PM-spaces will be used in the results of the main results.

2. Preliminaries

In an attempt to respond to many problems about imprecision in the natural world, it is appropriate to look upon the distance concept as a statistical rather than a determinate one. Guided by this, Menger [24], in 1942, introduced the notion of statistical metric spaces. In these spaces, the distance between points is a distribution function on \mathbb{R}^+ rather than a real number. Many authors studied such spaces (of special interest are books [25] by Schweizer and Sklar and [26] by Hadžić and Pap.)

We call function $F : \mathbb{R} \rightarrow [0, 1]$ a distribution function if it is a nondecreasing, left-continuous, and satisfies $F(0) = 0$ and $\sup_{x \in \mathbb{R}} F(x) = 1$. In the sequel, with ε_0 we will denote the specific distribution function defined by

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Fang et al. [16] defined the notion of an algebraic sum for two distribution functions.

Definition 5 ([16]). *Let $F, G \in D^+$. The algebraic sum of distribution functions F and G , in denotation $F \oplus G$, is defined by:*

$$(F \oplus G)(t) = \sup_{t_1+t_2=t} \min \{F(t_1), G(t_2)\}, \tag{1}$$

for every $t \in \mathbb{R}$.

From the previous definition, it is obvious that the following inequality

$$(F \oplus G)(t) \geq \min \{F(t_1), G(t_2)\} \tag{2}$$

holds for every $t > 0$, and arbitrary and fixed $t_1, t_2 > 0$, such that $t_1 + t_2 = t$.

Definition 6 ([25]). *A binary operation $T : [0, 1] \times [0, 1] \mapsto [0, 1]$ is continuous t -norm if T satisfies the following conditions:*

- (a) T is commutative and associative;

- (b) T is continuous;
- (c) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Examples of t -norm are $T(a, b) = \min\{a, b\}$ and $T(a, b) = ab$.

Definition 7. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (X, \mathcal{F}, T) where X is a nonempty set, T is a continuous t -norm, and \mathcal{F} is a mapping from $X \times X$ into D^+ ($\mathcal{F}(x, y) = F_{x,y}$ for every $(x, y) \in X \times X$) if and only if the following conditions hold:

- (PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
- (PM2) $F_{x,y}(t) = F_{y,x}(t)$;
- (PM3) $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$, for all $x, y, z \in X$ and $s, t \geq 0$.

Remark 1 ([15]). Every metric space is a PM-space. Let (X, d) be a metric space and $T(a, b) = \min\{a, b\}$ is a continuous t -norm. Define

$$F_{x,y}(t) = \varepsilon_0(t - d(x, y))$$

for all $x, y \in X$ and $t > 0$. The triple (X, \mathcal{F}, T) is a PM-space induced by the metric d .

Definition 8. Let (X, \mathcal{F}, T) be a Menger PM-space.

- (1) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists a positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.
- (2) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists a positive integer N , such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$.
- (3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X .

The (ε, λ) -topology in a Menger PM-space (X, \mathcal{F}, T) is introduced by Schweizer and Sklar [27]. Let \mathcal{N}_x be the family of neighborhoods of a point $x \in X$ given by

$$\mathcal{N}_x = \{N_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1)\}$$

where

$$N_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

Schweizer and Sklar [27] proved that the (ε, λ) -topology is a Hausdorff topology. Moreover, the following Lemmas are proven by Schweizer and Sklar [27].

Lemma 1 ([27]). If $p_n \rightarrow p$ then $F_{p,p_n}(t) \rightarrow F_{p,p}(t) = \varepsilon_0(t)$ for every $t > 0$, and conversely.

Lemma 2 ([27]). Let (X, \mathcal{F}, T) be a Menger PM-space and T is continuous. Then, the function \mathcal{F} is lower semi-continuous for every fixed $t > 0$, i.e., for every fixed $t > 0$ and every two convergent sequences $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subseteq X$, such that $x_n \rightarrow x, y_n \rightarrow y$, as $n \rightarrow \infty$, it follows that

$$\liminf_{n \rightarrow \infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Remark 2. In the previous, the Lemma continuity of T may be replaced by the weaker condition $\lim_{b \rightarrow 1} T(a, b) = a$.

The following lemma is a corollary of Lemma 2.

Lemma 3 ([25]). Let y be a fixed point and suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a convergent sequence such that $x_n \rightarrow x$, as $n \rightarrow \infty$. Then

$$\liminf_{n \rightarrow \infty} F_{x_n,y}(t) = F_{x,y}(t).$$

Definition 9. Let (X, \mathcal{F}, T) be a Menger PM-space and $A \subseteq X$. The closure of the set A is the smallest closed set containing A , denoted by \bar{A} .

Remark 3. Obviously $x \in \bar{A}$ if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in A such that $x_n \rightarrow x$, for $n \rightarrow \infty$.

The concept of probabilistic boundedness was defined by Egbert [28].

Definition 10. Let (X, \mathcal{F}, T) be a Menger PM-space and $A \subseteq X$. The probabilistic diameter of set A is given by

$$\delta_A(t) = \sup_{\varepsilon < t} \inf_{x,y \in A} F_{x,y}(\varepsilon).$$

The diameter of the set A is defined by

$$\delta_A = \sup_{t > 0} \delta_A(t).$$

If there exists $\lambda \in (0, 1)$ such that $\delta_A = 1 - \lambda$ the set A will be called probabilistic semi-bounded. If $\delta_A = 1$ the set A will be called probabilistic bounded.

Lemma 4 ([28]). Let (X, \mathcal{F}, T) be a Menger PM-space and $A \subseteq X$. Then $\delta_{\bar{A}} = \delta_A$ where \bar{A} denotes the closure of A in the (ε, λ) -topology.

Lemma 5 ([18]). Let (X, \mathcal{F}, T) be a Menger PM-space. A set $A \subseteq X$ is probabilistic bounded if and only if for each $\lambda \in (0, 1)$ there exists $t > 0$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in A$.

Remark 4. It is not difficult to see that every metrically bounded set is also probabilistic bounded if it is considered in the induced PM-space.

Sherwood [29] proved the analogue of Cantor’s theorem for Menger PM-spaces. In order to state this theorem, Sherwood [29] introduced the definition of probabilistic diameter which differs from the definition introduced by Egbert [28]. However, the following theorem remains true if we use Egbert’s definition of probabilistic diameter instead of Sherwood’s.

Theorem 1 ([29]). Let (X, \mathcal{F}, T) be a complete Menger PM-space and $\{F_n\}_{n \in \mathbb{N}}$ a nested sequence of nonempty, closed subsets of X such that $\delta_{F_n} \rightarrow \varepsilon_0$, for $n \rightarrow \infty$. Then, there is exactly one point $x_0 \in F_n$, for every $n \in \mathbb{N}$.

it is not difficult to prove that the following lemma holds.

Lemma 6 ([29]). Let (X, \mathcal{F}, T) be a complete Menger PM-space. Let $\{F_n\}_{n \in \mathbb{N}}$ be a nested sequence of nonempty, closed subsets of X . The sequence $\{F_n\}_{n \in \mathbb{N}}$ has a probabilistic diameter of zero, i.e., for each $\lambda \in (0, 1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in F_{n_0}$ if and only if $\delta_{F_n} \rightarrow \varepsilon_0$, for $n \rightarrow \infty$.

3. Main Results

Firstly, we will give a probabilistic version of the definition of R-weak commutativity of type A_g and of type A_f .

Definition 11. Let (X, \mathcal{F}, T) be a Menger PM-space. Two self-mappings f and g of X will be called R-weakly commuting of type A_g if there exists $R > 0$ such that $F_{ffx, gfy}(Rt) \geq F_{fx, gx}(t)$ holds for every $t > 0$. Similarly, two self-mappings f and g of X will be called R-weakly commuting of type A_f if there exists $R > 0$ such that $F_{fgx, ggy}(Rt) \geq F_{fx, gx}(t)$ holds for every $t > 0$.

The following lemmas are important for proving the main result.

Lemma 7 ([30]). If $\varphi: (0, \infty) \mapsto (0, \infty)$ is a continuous function that satisfies $\varphi(t) < t$ for every $t > 0$, then for every $t > 0$ we have that $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, where φ^n denotes the n -th iteration of φ .

Remark 5. The previous lemma is a corollary of Theorem 0.4, p. 21 from [30] and it is important for proving the main result.

Lemma 8. Let (X, \mathcal{F}, T) be a Menger PM-space. Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a continuous, function which satisfies $\varphi(t) < t$ for every $t > 0$. Then, the following statement holds:

If for $x, y \in X$ we have $F_{x,y}(\varphi(t)) \geq F_{x,y}(t)$ for every $t > 0$ then $x = y$.

Proof. Let us suppose the opposite, i.e., that $x \neq y$ holds for given assumption. By induction, we know that $F_{x,y}(\varphi^n(t)) \geq F_{x,y}(t)$ is satisfied. Letting $n \rightarrow \infty$, we obtain that $F_{x,y}(t) = 0$ for all $t > 0$, which is a contradiction with $\sup_{t>0} F_{x,y}(t) = 1$. \square

Lemma 9. Let (X, \mathcal{F}, T) be a Menger PM-space. If for two convergent sequences $\{x_n\}_{n \in \mathbb{N}}$, $\{y_n\}_{n \in \mathbb{N}} \subseteq X$ holds that $x_n \rightarrow p, y_n \rightarrow p$, as $n \rightarrow \infty$, then $F_{x_n, y_n}(t) \rightarrow 1$, as $n \rightarrow \infty$, for every $t > 0$.

Proof. Using condition (PM3) from Definition 7 it follows that

$$F_{x_n, y_n}(t) \geq T\left(F_{x_n, p}\left(\frac{t}{2}\right), F_{p, y_n}\left(\frac{t}{2}\right)\right)$$

holds for every $t > 0$. Letting $n \rightarrow \infty$ in previous inequality, using condition b) from Definition 6 and applying Lemma 1 we obtain that $F_{x_n, y_n}(t) \rightarrow 1$, for every $t > 0$. \square

Theorem 2. Let f and g be R -weakly commuting self-mappings of type A_f or of type A_g of a complete Menger PM-space (X, \mathcal{F}, T) , satisfying the condition

$$F_{gx, gy}(\varphi(t)) \geq \min \left\{ F_{fx, fy}(2t), F_{fx, gx}(t), F_{fy, gy}(t), (F_{fx, gy} \oplus F_{gx, fy})(\alpha t) \right\}, \tag{3}$$

for all $x, y \in X$, every $t > 0$ and every $\alpha > 3$, and for some continuous function $\varphi : (0, \infty) \rightarrow (0, \infty)$ which satisfies condition $\varphi(t) < t$, for every $t > 0$. Moreover, let $g(X)$ be a probabilistic bounded set and $g(X) \subseteq f(X)$. If mappings f and g are (f, g) -orbitally continuous then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary. From $g(X) \subseteq f(X)$ it follows that there exists a point $x_1 \in X$ such that $gx_0 = fx_1$. By induction, a sequence $\{x_n\}$ can be chosen such that $gx_{n-1} = fx_n$, for $n = 1, 2, \dots$

For sets $G_n = \{gx_n, gx_{n+1}, \dots\}$, $n \in \mathbb{N} \cup \{0\}$ let us prove that

$$\delta_{G_n}(\varphi(t)) \geq \delta_{G_{n-1}}(t) \tag{4}$$

holds for every $t > 0$ and every $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}_0$ be arbitrary. Since $F_{x,y}(\cdot)$ is a nondecreasing function, from conditions (1)–(3) (taking that $t_1 = t_2 = \frac{\alpha t}{2}$ and having in mind that $\frac{\alpha t}{2} > t$, for every $\alpha > 3$) we obtain that

$$\begin{aligned}
 F_{g^{x_{n+p}}, g^{x_{n+q}}}(\varphi(t)) &\geq \min \left\{ F_{f^{x_{n+p}}, f^{x_{n+q}}}(2t), F_{f^{x_{n+p}}, g^{x_{n+p}}}(t), F_{f^{x_{n+q}}, g^{x_{n+q}}}(t), \right. \\
 &\quad \left. (F_{f^{x_{n+p}}, g^{x_{n+q}}} \oplus F_{g^{x_{n+p}}, f^{x_{n+q}}})(\alpha t) \right\} \\
 &\geq \min \left\{ F_{g^{x_{n+p-1}}, g^{x_{n+q-1}}}(t), F_{g^{x_{n+p-1}}, g^{x_{n+p}}}(t), F_{g^{x_{n+q-1}}, g^{x_{n+q}}}(t), \right. \\
 &\quad \left. \sup_{t_1+t_2=\alpha t} \min \{ F_{g^{x_{n+p-1}}, g^{x_{n+q}}}(t_1), F_{g^{x_{n+p}}, g^{x_{n+q-1}}}(t_2) \} \right\} \\
 &\geq \min \left\{ F_{g^{x_{n+p-1}}, g^{x_{n+q-1}}}(t), F_{g^{x_{n+p-1}}, g^{x_{n+p}}}(t), F_{g^{x_{n+q-1}}, g^{x_{n+q}}}(t), \right. \\
 &\quad \left. F_{g^{x_{n+p-1}}, g^{x_{n+q}}}(t), F_{g^{x_{n+p}}, g^{x_{n+q-1}}}(t) \right\} \\
 &\geq \delta_{G_{n-1}}(t)
 \end{aligned}$$

holds for every $t > 0$ and every $n \in \mathbb{N}$. Finally, from Definition 10 and previous inequalities, it follows that

$$\delta_{G_n}(\varphi(t)) = \sup_{\varepsilon < \varphi(t)} \inf_{x, y \in G_n} F_{x, y}(\varepsilon) = \sup_{\varepsilon < \varphi(t)} \inf_{p, q \in \mathbb{N} \cup \{0\}} F_{g^{x_{n+p}}, g^{x_{n+q}}}(\varepsilon) \geq \delta_{G_{n-1}}(t)$$

holds, i.e., condition (4) holds for every $t > 0$ and every $n \in \mathbb{N}$.

From conditions (2) and (3), we obtain that

$$F_{g^{x, y}}(\varphi(t)) \geq \min \left\{ F_{f^{x, y}}(2t), F_{f^{x, g^x}}(t), F_{f^{y, g^y}}(t), F_{f^{x, g^y}}(2t), F_{f^{y, g^x}}(t) \right\} \tag{5}$$

holds for all $x, y \in X$, every $t > 0$, and for some continuous function $\varphi : (0, \infty) \rightarrow (0, \infty)$ which satisfies condition $\varphi(t) < t$, for every $t > 0$.

Now, we will prove that family $\{F_n\}_{n \in \mathbb{N}}$, where is $F_n = \overline{G_n}$, for every $n \in \mathbb{N}$, has probabilistic diameter zero. Let $\lambda \in (0, 1)$ and $t > 0$ be arbitrary. From the assumption that $g(X)$ is a probabilistic bounded set and $G_k \subseteq g(X)$, for arbitrary $k \in \mathbb{N}$, it follows that G_k is a probabilistic bounded set, also. Now, from Lemma 5 we have that for every $\lambda \in (0, 1)$ there exist $t_0 > 0$, such that

$$F_{x, y}(t_0) > 1 - \lambda \tag{6}$$

for all $x, y \in G_k$. Hence, for every $\lambda \in (0, 1)$ and such t_0 we obtain that

$$\delta_{G_k}(t_0) \geq 1 - \lambda.$$

From Lemma 7 it follows that there exists $l \in \mathbb{N}$ such that $\varphi^l(t_0) < t$. Then, it follows that $\varphi^{2m}(t_0) < t$ is satisfied for $m = \frac{p}{2}$, where p is an even number such that $p > l$. Let $n = 2m + k$ and $x, y \in G_n$ be arbitrary. Applying induction in (4) we obtain

$$\delta_{G_n}(t) \geq \delta_{G_n}(\varphi^{2m}(t_0)) \geq \delta_{G_{n-2m}}(t_0) \geq \delta_{G_k}(t_0) \geq 1 - \lambda$$

i.e.,

$$\delta_{G_n}(t) \geq 1 - \lambda.$$

From Lemma 4 sets G_n and F_n have the same probabilistic diameter. Then, we have that

$$\delta_{F_n}(t) \geq 1 - \lambda$$

i.e., we obtain that

$$F_{x, y}(t) \geq 1 - \lambda$$

for all $x, y \in F_n$, i.e., the family $\{F_n\}_{n \in \mathbb{N}}$ has a probabilistic diameter of zero.

From Theorem 1 and Lemma 6 we obtain that this family has nonempty intersection, which consists of exactly one point, z . Since the family $\{F_n\}_{n \in \mathbb{N}}$ has probabilistic diameter

zero and $z \in F_n$ for every $n \in \mathbb{N}$ then for every $r \in (0, 1)$ and every $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ holds

$$F_{gx_n, z}(t) > 1 - r.$$

Letting $r \rightarrow 0$, we obtain that

$$F_{gx_n, z}(t) \rightarrow 1,$$

as $n \rightarrow \infty$. Applying Lemma 1 we obtain $\lim_{n \rightarrow \infty} gx_n = z$. Now, from the definition of $\{fx_n\}$ it follows that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_{n-1} = z$. Finally, from (g, f) -orbitally continuity of mappings f and g we obtain

$$\lim_{n \rightarrow \infty} gfx_n = \lim_{n \rightarrow \infty} ggx_n = gz \quad \text{and} \quad \lim_{n \rightarrow \infty} ffx_n = \lim_{n \rightarrow \infty} fgx_n = fz. \tag{7}$$

From the assumption that f and g are R -weakly commuting of type A_g , there exists $R > 0$ such that

$$F_{ffx_n, gfx_n}(Rt) \geq F_{fx_n, gx_n}(t)$$

holds for every $t > 0$. Taking \liminf as $n \rightarrow \infty$ in previous inequality, then using Lemma 2 it follows that

$$F_{fz, gz}(Rt) \geq F_{z, z}(t) = 1$$

holds for such R and every $t > 0$. Hence, we obtain that $fz = gz$. \square

Remark 6. Analogous to the previous one, it can be proven that $fz = gz$ if the assumption is that f and g are R -weakly commuting mappings of type A_f .

Let us prove that point z is a common fixed point for mappings f and g . Then, from condition (5) we have

$$F_{gx_n, gz}(\varphi(t)) \geq \min \left\{ F_{fx_n, fz}(2t), F_{fx_n, gx_n}(t), F_{fz, gz}(t), F_{fx_n, gz}(2t), F_{fz, gx_n}(t) \right\},$$

for every $t > 0$. Having in mind that $fz = gz$, if we take \liminf as $n \rightarrow \infty$ in the previous inequality, then using Lemma 2 it follows that

$$\begin{aligned} F_{z, gz}(\varphi(t)) &\geq \min \left\{ F_{z, fz}(t), F_{z, z}(t), F_{z, fz}(t), F_{fz, gz}(t), F_{fz, z}(t) \right\} \\ &= \min \left\{ 1, F_{z, gz}(t) \right\} \\ &= F_{z, gz}(t) \end{aligned}$$

is satisfied, for every $t > 0$. From Lemma 8 it follows that $gz = z$. Hence z is a common fixed point of f and g .

Now, we will prove that z is a unique common fixed point. Suppose that $v \in X$ is another common fixed point for f and g , i.e., $fv = gv = v$. From (5) we obtain that

$$\begin{aligned} F_{z, v}(\varphi(t)) &= F_{gz, gv}(\varphi(t)) \\ &\geq \min \left\{ F_{fz, fv}(2t), F_{fz, gz}(t), F_{fv, gv}(t), F_{fz, gv}(2t), F_{fv, gz}(t) \right\} \\ &\geq \min \left\{ F_{z, v}(t), F_{z, z}(t), F_{v, v}(t) \right\} \\ &= \min \left\{ F_{z, v}(t), 1 \right\} \\ &= F_{z, v}(t). \end{aligned}$$

Finally, from Lemma 8 we obtain that $v = z$, i.e., z is a unique common fixed point for mappings f and g . This completes the proof.

Example 1. Let (X, \mathcal{F}, T) be a complete Menger probabilistic metric space induced by the metric $d(x, y) = |x - y|$ on $X = [1, 15] \subset \mathbb{R}$ given in Remark 1. We will prove that self-mappings $f, g : X \mapsto X$ defined by

$$f(x) = \begin{cases} 1, & x = 1, \\ 14, & 1 < x \leq 2, \\ \frac{3x-7}{13}, & x > 2 \end{cases} \quad g(x) = \begin{cases} 1, & x = 1, \\ 4, & 1 < x \leq 2, \\ 1, & x > 2 \end{cases}$$

have a unique common fixed point.

We will prove that the conditions of Theorem 2 are satisfied. Obviously, condition $g(X) \subseteq f(X)$ is satisfied. Now, we will prove that f and g are R -weakly commuting mapping of type A_f , i.e., we will prove that there exists $R > 0$ such that $F_{fgx,ggx}(Rt) \geq F_{fx,gx}(t)$ holds for every $t > 0$. Therefore, we will consider the following three cases:

Case 1 For $x = 1$ it follows that

$$F_{fgx,ggx}(Rt) = F_{1,1}(Rt) = 1 \geq F_{fx,gx}(t)$$

is satisfied for every $R > 0$ and $t > 0$.

Case 2 For every $1 < x \leq 2$ it is obvious that $ggx = 1, fgx = \frac{5}{13}, fx = 14$ and $gx = 4$. Then, from $F_{\frac{5}{13},1}(Rt) \geq F_{14,4}(t)$, i.e., $\epsilon_0(Rt - \frac{8}{13}) \geq \epsilon_0(t - 10)$ we obtain that f and g are R -weakly commuting mappings of type A_f for every $R \geq 1$, and every $t > 0$.

Case 3 For every $x > 2$ we obtain that f and g are R -weakly commuting mappings of type A_f for every $R > 0$ and every $t > 0$. Indeed, in this case we obtain that $F_{fgx,ggx}(Rt) = F_{1,1}(Rt) = 1$, for every $R > 0$ and every $t > 0$.

Let us define continuous function $\varphi : (0, +\infty) \mapsto (0, +\infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{1+t}, & 0 < t < \frac{1}{2}, \\ \frac{2}{3}t, & t \geq \frac{1}{2}. \end{cases}$$

This function satisfies condition $\frac{2}{3}t \leq \varphi(t) < t$, for every $t > 0$.

We will prove that the condition (3) is also satisfied. Hence, we will consider the following two cases:

Case 1 For $x = 1$ and for every $y \in (1, 2]$ it follows that

$$\begin{aligned} F_{gx,gy}(\varphi(t)) &= F_{1,4}(\varphi(t)) = \epsilon_0(\varphi(t) - 3) \geq \epsilon_0\left(\frac{2}{3}t - \frac{1}{3} \cdot 9\right) \\ &= \epsilon_0(2t - 9) \geq \epsilon_0(2t - d(fx, fy)) = F_{x,y}(2t) \\ &\geq \min \left\{ F_{fx,fy}(2t), F_{fx,gx}(t), F_{fy,gy}(t), (F_{fx,gy} \oplus F_{gx,fy})(\alpha t) \right\} \end{aligned}$$

holds for every $t > 0$ and every $\alpha > 3$.

Case 2 For $x \in (1, 2]$ and for every $y > 2$ it follows that

$$F_{gx,gy}(\varphi(t)) = F_{4,1}(\varphi(t)) = \epsilon_0(\varphi(t) - 3)$$

holds for every $t > 0$. Now, this case reduced to the Case 1.

All other possible cases are trivially satisfied. Moreover, it obvious that mapping f (mapping g) is (f, g) -orbitally continuous. Since all the conditions of Theorem 2 are satisfied, we have that $f(x)$ and $g(x)$ have a unique common fixed point. It is easy to see that this point is $x = 1$.

Now, we will prove the converse of the previous theorem for two R -weakly commuting self-mappings of type A_f or of type A_g . For that, we need to introduce one additional assumption.

Theorem 3. *Let the functions f and g satisfy all the assumptions of the Theorem 2 and let ggx_n converges for every sequence $\{x_n\}_{n \in \mathbb{N}}$ in X whenever gx_n converges. If mappings f and g have a unique common fixed point, then f and g are (g, f) -orbitally continuous.*

Proof. Let us assume that mappings f and g satisfy condition (3) and possess a common fixed point z . Moreover, (g, f) -orbit of any point x_0 defined by $gx_n = fx_{n+1}$, $n = 0, 1, 2, \dots$ converges to z , i.e., we have that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z.$$

Suppose that f and g are R -weakly commuting of type A_f . Then, there exists $R > 0$ such that $F_{fgx_n, ggx_n}(Rt) \geq F_{fx_n, gx_n}(t)$ holds, for every $t > 0$. Then, for such R and every $t > 0$ if we apply Lemma 9 it follows that

$$F_{fgx_n, ggx_n}(Rt) \rightarrow 1, \text{ as } n \rightarrow \infty. \tag{8}$$

Now, using condition (5) (with $x = gx_n$, $y = z$, and $gz = fz$) and condition (PM3) from Definition 7 we obtain

$$\begin{aligned} F_{ggx_n, gz}(\varphi(t)) &\geq \min \left\{ F_{fgx_n, gz}(2t), F_{fgx_n, ggx_n}(t), F_{gz, gz}(t), \right. \\ &\quad \left. F_{fgx_n, gz}(2t), F_{gz, ggx_n}(t) \right\} \\ &= \min \left\{ F_{fgx_n, gz}(2t), F_{fgx_n, ggx_n}(t), F_{gz, ggx_n}(t) \right\} \\ &\geq \min \left\{ T(F_{fgx_n, ggx_n}(t), F_{ggx_n, gz}(t)), F_{fgx_n, ggx_n}(t), \right. \\ &\quad \left. F_{gz, ggx_n}(t) \right\} \end{aligned} \tag{9}$$

Using the assumption that ggx_n converges for every sequence $\{x_n\}$ in X whenever gx_n converges, and having in mind condition (8) and conditions (b)–(d) from Definition 6, if we take \liminf as $n \rightarrow \infty$ in inequality (9) and apply it to Lemma 3 we obtain

$$F_{\lim_{n \rightarrow \infty} ggx_n, gz}(\varphi(t)) \geq F_{\lim_{n \rightarrow \infty} ggx_n, gz}(t).$$

Applying Lemma 8 we obtain that $\lim_{n \rightarrow \infty} ggx_n = gz$. Hence, g is (g, f) -orbitally continuous. Now, we will demonstrate that f is (g, f) -orbitally continuous. Indeed, using condition (PM3) from Definition 7, it follows that

$$F_{fgx_n, gz}(t) \geq T \left(F_{fgx_n, ggx_n} \left(\frac{t}{2} \right), F_{ggx_n, gz} \left(\frac{t}{2} \right) \right)$$

holds for every $t > 0$. Letting $n \rightarrow \infty$ in previous inequality, from condition (8) and Lemma 1 we obtain that $F_{fgx_n, gz}(t) \rightarrow 1$, for every $t > 0$. Finally, applying Lemma 1 we obtain $\lim_{n \rightarrow \infty} fgx_n = gz = fz$.

In a similar way, f and g are orbitally continuous if f and g are assumed R -weakly commuting of type A_g . This completes the proof. \square

4. Conclusions

In this paper, we proved that orbital continuity for a pair of self-mappings is a necessary and sufficient condition for the existence and uniqueness of a common fixed point for these mappings defined on Menger PM-spaces with a nonlinear contractive condition. The

main results are obtained using the notion of R -weakly commutativity of type A_f (or type A_g). Further research in this direction would be related to:

- Proving whether Theorems 2 and 3 remain true if we replace R -weakly commutativity of type A_f (or type A_g) with some other concept of commutativity of two self-mappings in the weaker sense. A positive answer for Theorem 2 in this sense was obtained by Ješić et al. [19] for a pair of semi R -commuting mappings;
- Proving whether Theorems 2 and 3 remain true if we replace the orbital continuity for two self-mappings with some other weakened condition of continuity.

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